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CHARACTERIZATION OF OPERATORS AND REVERBERATION CYCLES ASSOCIATED WITH A SINGLE NEURON EQUATION

by

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Introduction

Since E.R.Caianiello [3] proposed a mathematical model of neural behaviors, numerous contributions have been given by himself and his collaborators regarding mathematical properties of the neural equations. Among these contributions the reverberation phenomenon of a neural network has been one of their main concerns. For instsnce they investigated neural networks whose reverberation cycles are required not to exceed a preassigned maximal length, as shown in Caianiello and others [1]~[9], [11]~[16]. Their main mathematical tecniques rely upon matrix algebra.

On the other hand Kitagawa[10] recently started with discussion of dynamical behaviors of all the possible solutions of the neural equation for the case of single neuron. Among others, Kitagawa [10] introduced various specal operators and investigated some characteristic features of the dynamics led by each of these operators. This paper is especially connected with detailed discussions of these operators introduced by Kitagawa [10].

In Section 1 we shall give a necessary and sufficient condition for that all the sequences obtained from the solutions of the neural equation (1.1) form a set of mutually disjoint reverberation cycles and show that the neural system satisfying the above condition is equivalent to that under either of two operators L_{α_0} and $L_{\overline{\alpha_0}}$ introduced by Kitagawa [10]. In section 2 we then define a digraph with respect to each of operators which were introduced by Kitagawa[10] in connection with (1.1). Our grapf theoretical consideration shows that there exists a set of 2(n+1) special operators, called an α-ω set of operators, in te terms of which any operator associated with (1.1) can be represented in a unique way. In Section3, we deal with reverberation cycles associated with each of these fundamental operators L_{α_ℓ} and $L_{\overline{\alpha_\ell}}$ (ℓ =0,1,...,n-1). We determine the exact number of all the possible reverberation cycles with any admissible length. Here we make use oe a combinatrial method appealing to circular partitions. We then proceed to discuss the reverberation phenomena for operators L_ℓ and $L_{\overline{\alpha_\ell}}$ as well. It is shown that the number of all the reverberation clcles can be reduced to those of reverberation cycles under the operators L_{α_0} and $L_{\bar{\alpha}_0}$.

§1. Characterization of the operators L_{α_0} and $L_{\bar{\alpha}_0}$

We shall investigate mathematical properties of a single neuron model represented by a nonlinear difference equation which read :

(1.1) $$x(t+1) = 1\left[\sum_{k=0}^{n-1} a_k x(t-k) - \theta\right]$$

(1.2) $$1[u] = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u \leqq 0, \end{cases}$$

where x(t) is the state (1 or 0) of a neuron at the time t in the set I of indices $(0, \pm 1, \pm 2, \ldots)$, a_k is the coupling coefficient from the neuron to itself which is effective k unit times after firing, and θ is the threshold value.

In what follows in this paper we assume thet all a_k $(k = 0, 1, \ldots, n-1)$ and θ are constant independently of t. The functional equation (1.1) is regarded a neural equation (NE) due to Caianiello [1] for a single neuron.

In order to investigate state transition phenomena of (1.1), it is convinient to introduce the following n-dimentional vector as shown in Kitagawa [10]:

(1.3) $$\boldsymbol{\delta}_t = (\delta_t, \delta_{t-1}, \cdots, \delta_{t-n+2}, \delta_{t-n+1})$$

where $\delta_s \equiv x(s)$ and δ_t denotes the state of our neuron at time t, while $\boldsymbol{\delta}_t$ is called an n-state configuration of our neuron at time t. Let us define the inner product $(\mathbf{a}, \boldsymbol{\delta}_t)$ by

(1.4) $$(\mathbf{a}, \boldsymbol{\delta}_t) \equiv \sum_{k=0}^{n-1} a_k \delta_{t-k}$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and a concurrence function by

(1.5) $$Z(\boldsymbol{\delta}_t) \equiv (\mathbf{a}, \boldsymbol{\delta}_t) - \theta,$$

which reduces the functional equation (1.1) to

(1.6) $$\delta_{t+1} = 1[Z(\boldsymbol{\delta}_t)].$$

In connection with (1.6) Kitagawa [10] introduced the operator L of n-dimentional vectors defined by

(1.7) $$L(\delta_t, \delta_{t-1}, \cdots, \delta_{t-n+2}, \delta_{t-n+1}) = (1[\Sigma(\delta_t)], \delta_t, \cdots, \delta_{t-n+2})$$

The five characteristic features of the functional equation of (1.1) have been pointed out by Kitagawa[10], one of which is the translatability of associated operators. An operator L defined by (1.7), namely

(1.8) $$L(\delta_t) = (\Lambda\delta_t, \delta_t, \delta_{t-1}, \cdots, \delta_{t-n+2})$$

with

(1.9) $$\Lambda\delta_t = 1[\Sigma(\delta_t)]$$

is translatable, i.e., commutative with all translations T_α, $-\infty<\alpha<\infty$, where $T_\alpha z(t) \equiv z(t+\alpha)$, in the sense that we have

(1.10) $$T_\alpha(L(\delta_t)) = L(T_\alpha(\delta_t))$$

for all t and α. Because of this translatability property, we may and we shall confine our discussion to a transition :

(1.11) $$L(\delta_{n-1}, \cdots, \delta_1, \delta_0) = (1[\Sigma(\delta)], \delta_{n-1}, \cdots, \delta_1)$$

for any $\delta = (\delta_{n-1}, \cdots, \delta_0)$ in the n-state configuration space X_n, which is defined by

(1.12) $$X_n = \{\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0);\ \delta_i = 1 \text{ or } 0,\ i = 0, 1, \cdots, n-1\}.$$

Now we shall give the following

Definition 1.1. A solution x(t) of the functional equation (1.1) is said to have a reverberation cycle, abbreviated by RVC, of period R if the following equality holds:

(1.13) $$X(t+R) = x(t)$$

for any $t \in I = \{0, \pm 1, \pm 2, \ldots\}$.

Let us define $L^i(\delta) \equiv L(L^{i-1}(\delta))$ and $L^0(\delta) \equiv \delta$.

Lemma 1.1. The equality (1.13) holds if and only if the following equality holds :

(1.14) $$L^R(\delta_t, \delta_{t-1}, \cdots, \delta_{t-n+2}, \delta_{t-n+1}) = (\delta_t, \delta_{t-1}, \cdots, \delta_{t-n+2}, \delta_{t-n+1})$$

for any $t \in I$.

Proof. The proof is trivial.

Because of Lemma 1.1, our search for any reverberation cycle is equivalent to that for the solutions of equation (1.7) with the condition (1.14).

For the operator L, an immediate observation gives the following

Lemma 1.2. For any assigned n-state configuration in X_n the number of state configurations transformed to δ by an application of L is at most two. In particular, when any assigned δ belongs to some reverberation cycle there is one and only one δ' such that $L(\delta') = \delta$.

Kitagawa[10] was concerned with the following special operator;

$$L_{\bar{\alpha}_0}(\delta_{n-1}, \cdots, \delta_1, \delta_0) \equiv (\bar{\delta}_0, \delta_{n-1}, \cdots, \delta_1) \tag{1.15}$$

for any state configuration $(\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n and showed that for any state configuration there exists a positive integer R such that equality (1.14) holds, that is, any n-state configuration in X_n belongs to some reverberation cycle, under the operator $L_{\bar{\alpha}_0}$. Here we shall introduce the another special operator

$$L_{\alpha_0}(\delta_{n-1}, \cdots, \delta_1, \delta_0) \equiv (\delta_0, \delta_{n-1}, \cdots, \delta_1) \tag{1.16}$$

for any $(\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n.

In connection with these operators $L_{\bar{\alpha}_0}$ and L_{α_0} just introduced, it is important to oberve that there exist sets of coupling coefficients $\{a_k\}$ and the threshold value θ for any of which the operator L associated with the functional equation (1.1) is equivalent to one of these two operators L_{α_0} and $L_{\bar{\alpha}_0}$

Lemma 1.3. There exist sets of coupling coefficients $\{a_i;\ i=0,1,\cdots, n-1\}$ of and the threshold value θ such that (i) $l[z(\delta)] = \bar{\delta}_0$ for any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n or (ii) $l[z(\delta)] = \delta_0$ for any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n.

Proof. The case (i). For any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n, let us assume $l[z(\delta)] = \bar{\delta}_0$ which holds if and only if the following inequalities hold:

$$a_{n-1} + \sum_{j=1}^{h} a_{i_j} - \theta \leqq 0 \tag{1.17}$$

$$\sum_{j=1}^{h} a_{i_j} - \theta > 0, \tag{1.18}$$

for any set $(i_1, i_2, \quad, i_h)$, $0 \leq h \leq n-1$, such that $0 \leqq i_1 < i_2 < i_h \leqq n-2$. For any fixed a_i, $i=0,1,2,\cdots,n-2$, let us put

$$A^{(+)} \equiv \sum_{a_i > 0} a_i \qquad i = 0, 1, \cdots, n-2 \tag{1.19}$$

(1.20) $$A^{(-)} \equiv \sum_{a_i<0} a_i \qquad i=0,1,\cdots,n-2$$

It is trivial that (1.17) and (1.18) for a_{n-1} and θ hold true if

(1.21) $$a_{n-1} + A^{(+)} \leqq \theta < A^{(-)}.$$

The case (ii). This is quite similar to the case (i).

Under each of these operators $L_{\bar{\alpha}_0}$ and L_{α_0} any solution of (1.1) is dependent on the sole state $\sigma_{t-(n-1)}$ exactly n step before the time t.

Now we shall prove the following

Theorem 1.1. The necessary and sufficient condition for that any n-state configuration $\delta=(\delta_{n-1},\cdots,\delta_1,\delta_0)$ in X_n belongs to some reverberation cycle is that

$L=L_{\bar{\alpha}_0}$ or L_{α_0}, that is, $1[z(\delta)]=\bar{\delta}_0$ or δ_0.

Proof. (1) Necessity. For any $\tilde{\delta}=(\delta_{n-1},\cdots,\delta_1)$, let us put

(1.22)

(i) $$\delta^{(0)}=(\delta_{n-1},\cdots,\delta_1,0)=(\tilde{\delta},0),$$

(ii) $$\delta^{(1)}=(\delta_{n-1},\cdots,\delta_1,1)=(\tilde{\delta},1),$$

(iii) $$\min_{\tilde{\delta}\in X_{n-1}} z(\delta^{(1)})\equiv l$$

(iv) $$\max_{\tilde{\delta}\in X_{n-1}} z(\delta^{(1)})\equiv u.$$

Now it is evident that the following three cases exhaust all the possibilities:

Case 1. $l>0$

Case 2. $u\leqq 0$

Case 3. $l\leqq 0<u$.

Case1. In this case, for any $\tilde{\delta}=(\delta_{n-1},\cdots,\delta_1)$, $z(\delta^{(1)})>0$ and hence we have

(i.23) $$1[z(\delta^{(1)})]=1,$$

which implies

(1.24) $$l\,[\,z\,(\delta^{(0)})\,] = 0,$$

because $l[z(\delta^{(0)})] = 1$, i.e., $L(\delta^{(0)}) = L(\delta^{(1)})$ contradicts Lemma 1.2.

From (1.23) and (1.24), we have $L\,(\delta_{n-1}, \cdots, \delta_1, \delta_0) = (\delta_0, \delta_{n-1}, \cdots, \delta_1)$ for any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n, which means that L is equivalent to the operator L_{α_0}.

Case 2. In this case for any $\tilde{\delta} = (\delta_{n-1}, \cdots, \delta_1)$, $z\,(\delta^{(1)}) \leqq 0$, and hence we have

(1.25) $$l\,[\,z\,(\delta^{(1)})\,] = 0,$$

which implies, in view of Lemma 1.2,

(1.26) $$l\,[\,z\,(\delta^{(0)})\,] = 1$$

as in Case 1.

Frome (1.25) and (1.26), we have $L(\delta_{n-1}, \cdots, \delta_1, \delta_0) = (\delta_0, \delta_{n-1}, \cdots, \delta_1)$ for any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n, which implies that L is equivalent to the operator $L_{\overline{\alpha}_0}$.

Case 3. Now let us introduce an (n-1)-state configuration

(1.27) $$\tilde{\delta}^{(-)} = (\delta_{n-1}^{(-)}, \cdots, \delta_1^{(-)}),$$

whreye

(1.28) $$\delta_j^{(-)} = \begin{cases} 1 & \text{if } a_j < 0 \\ 0 & \text{if } a_j \geqq 0 \end{cases} \qquad j = 1, 2, \cdots, n-1,$$

and another one

(1.29) $$\tilde{\delta}^{(+)} = (\delta_{n-1}^{(+)}, \cdots, \delta_1^{(+)}),$$

where

(1.30) $$\delta_j^{(+)} = \begin{cases} 1 & \text{if } a_j > 0 \\ 0 & \text{if } a_j \leqq 0 \end{cases} \qquad j = 1, 2, \cdots, n-1 \quad .$$

Then it is clear that

(1.31) $$\ell = z\,((\tilde{\delta}^{(-)}, 1))$$

(1.32) $$u = z\,((\tilde{\delta}^{(+)}, 1)).$$

Now we shall prove that the Case 3 does not occur by ad sbsurde.

Let $\ell \leqq 0$. Then it follows that

(1.33) $$l\,[\,z\,(\tilde{\delta}^{(-)}, 1)\,] = 0$$

which implies, in view of Lemma 1.2,

(1.34) $$l\,[\,z\,(\tilde{\delta}^{(-)}, 0)\,] = 1,$$

which is equivalent to

(1.35) $\quad \Sigma((\hat{\delta}^{(-)}, 0)) > 0.$

From (1.33) and (1.35) we have

(1.36) $\quad \Sigma((\hat{\delta}^{(-)}, 1)) - \Sigma((\hat{\delta}^{(-)}, 0)) = a_{n-1} < 0.$

On the other hand let $u > 0$. Then it follows that

(1.37) $\quad 1\,[\Sigma(\hat{\delta}^{(+)}, 1)] = 1,$

which implies, in view of Lemma 1.2.

(1.38) $\quad 1\,[\Sigma(\hat{\delta}^{(+)}, 0)] = 0,$

which is equivalent to

(1.39) $\quad \Sigma((\hat{\delta}^{(+)}, 0)) \leqq 0.$

From (1.37) and (1.39) we have

(1.40) $\quad \Sigma((\hat{\delta}^{(+)}, 1)) - \Sigma((\hat{\delta}^{(+)}, 0)) = a_{n-1} > 0.$

But (1.36) and (1.40) are mutually contradictory, showing that the Case 3 does not exist.

(2) Sufficiency. This is trivial, because, for any δ in X_n we have

(1.41) $\quad L_{\bar{\alpha}_0}^{2n}(\delta) = \delta$

(1.42) $\quad L_{\alpha_0}^{n}(\delta) = \delta,$

which show that any δ in X_n belongs to some RVC of at most length 2n ~~and~~ respectively under successive applications of $L_{\bar{\alpha}_0}$ and those of L_{α_0}, respectively. q.e.d.

Our Lemma 1.3 and Theorem 1.1 give us a characteristic of L_{α_0} and $L_{\bar{\alpha}_0}$ in the realm of functional operators defined by (1.11).

§2. Graphical representation of the operator L

In this Section a digraph G_L is defined for each operator L introduced by (1.11), and we shall represent G_L in terms of a set of specific operators which includes L_{α_0} and $L_{\bar{\alpha}_0}$ as its members.

Definition 2.1. A digraph $G = (X_n, \Gamma)$ is called to be a digraph with respect to the operator L when the set of arcs is given by $\Gamma = \{(\delta, L(\delta))\,;\ \delta \in X_n\}$. In this case we denote the digraph by $G_L = (X_n, \Gamma_L)$.

Let us define the following operations between two digraphs:

Définition 2.2. For any two digraphs $G = (X, \Gamma)$ and $G' = (X, \Gamma')$,

(2.1) $\quad G + G' = (X, \Gamma \cup \Gamma')$

(2.2) $\quad G.G = (X, \Gamma \cap \Gamma').$

Definition 2.3. The total number of arcs in a digraph (X_n, Γ)

which starts from an assigned δ in X_n is called to be the outdegree of δ in the digraph (X_n, Γ).

Definition 2.4. The number of arcs in a digraph (X_n, Γ) which comes to an assigned δ in X_n is called to be the indegree of δ in the digraph (X_n, Γ).

From the definition of Γ_L and Lemma 1.2 we have immediately the following

Corollary 2.1. For a digraph $G_L = (X_n, \Gamma_L)$ w.r.t. an operator L, we have

(2.3) outdegree of $\delta = 1$

(2.4) indegree of $\delta \leqq 2$,

for any $\delta \in X_n$. In particular any δ belongs to some reverberation cycle if and only if indegree of $\delta = 1$.

Definition 2.5. A digraph $G = (X_n, \Gamma)$ is called to be $\mathcal{L}$-complete if, any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ contains two arcs $(\delta, (1, \delta_{n-1}, \cdots, \delta_1))$ and $(\delta, (0, \delta_{n-1}, \cdots, \delta_1))$.

Definition 2.6. A digraph $G = (X_n, \Gamma)$ is called to be $\mathcal{L}$-complement to a digraph $G_L = (X_n, \Gamma_L)$ w.r.t. an operator L if $G + G_L$ is $\mathcal{L}$-complete, denoted by $G = G_L^c$.

Lemma 2.1. For any operator L, there exists one and only one operator $\bar{L}$ such that a digraph $G_{\bar{L}}$ w.r.t. the operator $\bar{L}$ is an $\mathcal{L}$-complement to the digraph G_L w.r.t. the operator L.

Proof. For any operator L there exist the set $\{a_i;\ i=0,1, \cdots, n-1\}$ of coupling coefficients and threshold value θ corresponding to the operator L. Because of density of the set of real numbers, we may and we shall confine to the set $\{a_i;\ i=0,1, \cdots, n-1\}$ and θ such that exact inequalities hold

$$(2.5)\qquad 1[\Sigma(\delta; \mathfrak{a}, \theta)] = \begin{cases} 1 & \text{if } \Sigma(\delta; \mathfrak{a}, \theta) > 0 \\ 0 & \text{if } \Sigma(\delta; \mathfrak{a}, \theta) < 0, \end{cases}$$

where $\Sigma(\delta; \mathfrak{a}, \theta) = (\mathfrak{a}, \delta) - \theta$ for $\mathfrak{a} = (a_0, a_1, \cdots, a_{n-1})$ and any δ in X_n.

Now let us introduce the operator $\bar{L}$ corresponding to the set $\{\bar{a}_i;\ i=0,1,\cdots,n-1\}$ and threshold value $\bar{\theta}$, where $\bar{a}_i \equiv -a_i$ for $i = 0, 1, \cdots, n-1$ and $\bar{\theta} = -\theta$. Then for any δ in X_n

(2.6) $\Sigma(\delta; \mathfrak{a}, \theta) \gtrless 0$ if and only if $\Sigma(\delta; \bar{\mathfrak{a}}, \bar{\theta}) \lessgtr 0$ where $\bar{\mathfrak{a}} = (\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{n-1})$.

(2.6) is equivalent to say that, for any δ, (2.7) $\mathcal{L}(\delta_{n-1}, \cdots, \delta_1, \delta_0) = (\delta, \delta_{n-1}, \cdots \delta_1)$

implies and is implied by

(2.8) $\bar{\mathcal{L}}(\delta_{n-1}, \cdots, \delta_1, \delta_0) = (\bar{\delta}, \delta_{n-1}, \cdots, \delta_1)$.

This means that $G_L + G_{\bar{L}}$ is $\mathcal{L}$-complete, this is to say, $G_{\bar{L}}$ is $\mathcal{L}$-complement to G_L. The uniqueness of the operator L is trivial. q.e.d.

Lemma 2.2. For any operator L, we have the equation

(2.9) $G^{*} \cdot G_{\mathcal{L}} = G_{\mathcal{L}} \cdot G^{*} = G_{\mathcal{L}},$

(2.10) $G^{*} + G_{\mathcal{L}} = G_{\mathcal{L}} + G^{*} = G^{*}.$

where G* is the $\mathcal{L}$-complete digraph.

Proof. This is evident from the definition of G*.

Now let us consider a system of operators consististing $\{L_{\alpha_\ell}\}, \{L_{\bar{\alpha}_\ell}\}$ $(\ell = 0, 1, \cdots, n-1)$, $\mathcal{L}_\omega$ and $\mathcal{L}_{\bar{\omega}}$ which are introduced in KItagawa [10] and defined in the following way:

(2.11) $\mathcal{L}_{\bar{\alpha}_\ell}(\delta_{n-1}, \cdots, \delta_\ell, \cdots, \delta_0) \equiv (\bar{\delta}_\ell, \delta_{n-1}, \cdots, \delta_1)$ $\quad \ell = 0, 1, \cdots, n-1$

(2.12) $\mathcal{L}_{\alpha_\ell}(\delta_{n-1}, \cdots, \delta_\ell, \cdots, \delta_0) \equiv (\delta_\ell, \delta_{n-1}, \cdots, \delta_1)$

(2.13) $\mathcal{L}_{\bar{\omega}}(\delta_{n-1}, \cdots, \delta_1, \delta_0) \equiv (0, \delta_{n-1}, \cdots, \delta_1)$

(2.14) $\mathcal{L}_\omega(\delta_{n-1}, \cdots, \delta_1, \delta_0) \equiv (1, \delta_{n-1}, \cdots, \delta_1),$

for any $\delta = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$ in X_n. The set of all these operators is called to be the $\alpha - \omega$ set of operators in X_n.

Similarly as in Lemma 1.3 we obtain

Lemma 2.3. There exist the sets of coupling coefficients $\{a_i ; i = 0, 1, \cdots, n-1\}$ and threshold value θ corresponding to each of the operators defined by (2.11), (2.12), (2.13) and (2.4), respectively.

Lemma 2.4. We have the equations

(2.15) $G_{\mathcal{L}_{\alpha_\ell}} + G_{\mathcal{L}_{\bar{\alpha}_\ell}} = G^{*}$

(2.16) $G_{\mathcal{L}_\omega} + G_{\mathcal{L}_{\bar{\omega}}} = G^{*},$

for $\ell = 0, 1, \cdots, n-1$. where G* is the $\mathcal{L}$-complete graph.

Proof. This is evident from the definition of each operators.

From Lemma 2.4, we have immediately the following

Corollary 2.2. We have relations

(2.17) $G_{\mathcal{L}_{\alpha_\ell}} = G^{c}_{\mathcal{L}_{\bar{\alpha}_\ell}}$

(2.18) $G_{\mathcal{L}_\omega} = G^{c}_{\mathcal{L}_{\bar{\omega}}},$

for $\ell = 0,1, \cdots , n-1$.

Corollary 2.3. We have relations

(2.19) $$G_{L_{\alpha_\ell}} \cdot G_{L_{\bar{\alpha}_\ell}} = G_\phi$$

(2.20) $$G_{L_\omega} \cdot G_{L_{\bar{\omega}}} = G_\phi,$$

for ℓ = = $o,1, \cdots , n-1$, where G_ϕ denote a special digraph when the set of arcs reduced to an empty set ϕ .

Any operator L defines and is defined by δ such that $1[z(\delta)] = \delta$ because we have

(2.21) $L(\delta) = (1[z(\delta)], \delta_{n-1}, \cdots \delta_1) = (\delta_0, \delta_{n-1}, \cdots, \delta_1)$.

This fact makes us possible to introduce the notion

(2.22) $(\delta ; \delta) \in L$.

Now a representation of any operator L in X_n on the basis the set of operators by the help of the notion (2.22) is given by the following

Theorem 2.1. A digraph G_L w.r.t. any assigned operator L in X_n is represented by

(2.23) $$G_L = \sum_{\substack{(\delta ; \delta) \in L \\ \delta = (\delta_{n-1}, \cdots, \delta_0) \in X_n}} G(\delta) \prod_{\ell=0}^{n-1} G(\delta, \delta_\ell),$$

where digraphs $G(\delta, \delta_\ell)$ and $G(\delta)$ are defined by

(2.24) $$G(\delta, \delta_\ell) = \begin{cases} G_{L_{\alpha_\ell}} & \text{if } \delta = \delta_\ell \\ G_{L_{\bar{\alpha}_\ell}} & \text{if } \delta = \bar{\delta_\ell} \end{cases}$$

(2.25) $$G(\delta) = \begin{cases} G_{L_\omega} & \text{if } \delta = 1 \\ G_{L_{\bar{\omega}}} & \text{if } \delta = 0. \end{cases}$$

Proof. Let us denote any arc in digraph G_L by

(2.26) $((\delta_{n-1}, \cdots \delta_1, \delta_0), (\delta, \delta_{n-1}, \cdots \delta_1)) = (\delta, \delta'),$

which implies and is implied by $(\delta ; \delta) \in L$. By the definitions of the digraphs $G(\delta, \delta_\ell)$ and $G(\omega)$, we have

(2.27) $(\delta, \delta') \in G(\delta, \delta_\ell),$

for ℓ = $o,1, \cdots , n-1$, and also

(2.28) $(\delta, \delta') \in G(\delta),$

which give us

(2.29) $$(\delta, \delta') \in G(\delta) \prod_{l=0}^{n-1} G(\delta, \delta_l).$$

Hence we have

(2.30) $$G_{\angle} \subseteq \sum_{\substack{(\delta;\, \delta')\in \angle \\ \delta' = (\delta_{n-1}, \cdots, \delta_0) \in X_n}} G(\delta) \prod_{l=0}^{n-1} G(\delta, \delta_l).$$

On the other hand, for any $(\delta;\, \delta') \in \angle$ we have

(2.31) $$G(\delta) \prod_{l=0}^{n-1} G(\delta, \delta_l) = ((\delta_{n-1}, \cdots, \delta_1, \delta_0), (\delta, \delta_{n-1}, \cdots, \delta_1)),$$

where $\delta' = (\delta_{n-1}, \cdots, \delta_1, \delta_0)$, because the set of arcs in the digraphs defined by

(2.32) $$\prod_{l=0}^{n-1} G(\delta, \delta_l)$$

consists of two arcs

(2.33) $$((\delta_{n-1}, \cdots, \delta_1, \delta_0), (\delta, \delta_{n-1}, \cdots, \delta_1))$$

(2.34) $$((\overline{\delta_{n-1}}, \cdots, \overline{\delta_1}, \overline{\delta_0}), (\overline{\delta}, \overline{\delta_{n-1}}, \cdots, \overline{\delta_1})),$$

to which the multiplication of digraph $G(\delta)$ leads to (2.31).

Equation (2.30) together with (2.31) gives us (2.23), as we were to prove. q.e.d.

§3. The number of reverberation cycles

In this Section we shall deal with the α-ω set of operators defined in Section 2 and obtain the number of all reverberation cycles with possible length.

We shall define a notion of a circular partition for any positive integer. The total number of circular partitions is then given by virtue of the number of ordered partitions each of which is usually called composition after P.A.Macmahon. So far as our enumerations of reverberation cycles are concered, the advantage of the notion of circular of partitions is that it makes us possible to introduce the notion of equivalent classes in the set of all ordered partitions.

A partition of a positive integer n is a represented by a sum of positive integers as follows :

(3.1) $$n = t_1 + t_2 + \dots + t_k ,$$

where $t_i \geqq 1$, $1 \leqq i \leqq k$,for $k = 1,2,\dots,n$, Eq. (3.1) is called a k-partition of n and denoted by $(t_1, t_2, \dots, t_k)$. An ordered k-partition is called by k-ordered partition. It is conventional to abbreviate repeated parts, by use of exponents ; for example, 6-ordered partition (2,3,2,3,2,3) of n=15 is written $(2,3)^3$. A special permutation σ of $(t_1,t_2,\dots,t_k)$ is defined by

(3.2) $$\sigma(t_1, t_2, \cdots, t_k) = (t_k, t_1, t_2, \cdots, t_{k-1})$$

and let us put

(3.3) $$\sigma^j(t_1, t_2, \cdots, t_k) = \sigma(\sigma^{j-1}(t_1, t_2, \cdots, t_k))$$

(3.4) $$\sigma^0(t_1, t_2, \cdots, t_k) = (t_1, t_2, \cdots, t_k),$$

for $j \geqq 1$.

Definition 3.1. Two k-ordered partitions, $(t_1,t_2,\dots,t_k)$ and $(s_1,s_2,\dots,s_k)$, of n are called to be equivalent if there exists a positive integer j ($1 \leqq j \leqq k$) such that

(3.5) $$\sigma^j(t_1, t_2, \cdots, t_k) = (s_1, s_2, \cdots, s_k).$$

Let us denote by $P_{n,k}$ the set of all the possible k-ordered partitions of n and by $C_{n,k}$ the set of all the possible equivalent classes of $P_{n,k}$, each element of which is called a k-circular partitions of n. The equivalent class containing the k-ordered partition $(t_1,t_2,\dots,t_k)$ is denoted by $E(t_1,$

$t_2,\ldots,t_k$). In what follows we shall denote by $\#A$ the number of all the elements belonging to a set A.

Lemma 3.1. For any k-ordered partition $(t_1,t_2,\ldots,t_k)$ of n, for which $\#E(t_1,t_2,\ldots,t_k) = j$ holds, there is a positive integer d for which we have

$$(3.6)\qquad \begin{cases} (1)\quad (t_1,t_2,\cdots,t_k) = (t_1,t_2,\cdots,t_j)^d, \text{ with } dj=k, \\ (2)\quad \#E(t_1,t_2,\cdots,t_j) = j, \\ (3)\ \text{d is a divisor of g.c.m. of n and k, denoted by } d \mid (n,k). \end{cases}$$

Proof. Because of $\sigma^k(t_1,t_2,\ldots,t_k) = (t_1,t_2,\ldots,t_k)$, we have

$$(3.7)\qquad 1 \leqq \#E(t_1,t_2,\cdots,t_k) \leqq k.$$

It is evident that (3.6) is valid for j=1 or k. Now let us assume that $1<j<k$. This implies that there exists a uniquely determined positive integer h $(1<h<k)$ scch that

$$(3.8)\qquad \sigma^h(t_1,t_2,\cdots,t_k) = (t_1,t_2,\cdots,t_k)$$

$$(3.9)\qquad \sigma^i(t_1,t_2,\cdots,t_k) \neq (t_1,t_2,\cdots,t_k) \quad \text{for } i=1,2,\cdots,h-1.$$

It is clear that h is a divisor of k, because the equation

$$(3.10)\qquad \sigma^{sh+r}(t_1,t_2,\cdots,t_k) = \sigma^r(t_1,t_2,\cdots,t_k)$$

with $k=sh+r$, for $0\leqq r<h-1$, gives us $r=0$. Hence any k-ordered partition of $E(t_1,t_2,\ldots,t_k)$ can be written as follows :

$$(3.11)\qquad \sigma^{ph+i}(t_1,t_2,\cdots,t_k) = \sigma^i(t_1,t_2,\cdots,t_k),$$

for $p=0,1,\ldots,k/d$ and $i=0,1,\ldots,h-1$, which implies $h=j$. In consequence we have

$$(3.12)\qquad (t_1,t_2,\cdots,t_k) = (t_1,t_2,\cdots,t_j)^d$$

with $dj=k$. The equality $d\sum_{i=1}^{j} t_i=n$ gives us $d \mid n$ which implies, in view of $dj=k$, $d \mid (n,k)$. q.e.d.

Let us put

$$(3.13)\qquad \psi(n,k) = \#\{(t_1,t_2,\cdots,t_k) : (t_1,t_2,\cdots,t_k) \in P_n,\ \#E(t_1,t_2,\cdots,t_k) = k\}.$$

We shall give the following

Lemma 3.2. For any positive integer n and k, $1\leqq k\leqq n$, we have

$$(3.14)\qquad \sum_{d\mid(n,k)} \psi(n/d,\ k/d) = \binom{n-1}{k-1}.$$

Proof. It is trivial that for any positive integer n and k, $1\leqq k\leqq n$, we have

(3.15) $$\# P_{n,k} = \binom{n-1}{k-1}.$$

On the other hand, we have, from Lemma 3.1,

(3.16) $$\# P_{n,k} = \sum_{d|(n,k)} \#\{(t_1,\cdots,t_k)\,;\,(t_1,\cdots,t_k)\in P_{n,k},\ \#E(t_1,\cdots,t_k)=k/d\}$$

and

(3.17) $$\begin{aligned}&\#\{(t_1,\cdots,t_k)\,;\,(t_1,\cdots,t_k)\in P_{n,k},\ \#E(t_1,\cdots,t_k)=\tfrac{k}{d}\}\\ &=\#\{(t_1,\cdots,t_{k/d})\,;\,(t_1,\cdots,t_{k/d})\in P_{n/d,k/d},\ \#E(t_1,\cdots,t_{k/d})=\tfrac{k}{d}\}\\ &=\psi(n/d,k/d)\end{aligned}$$

for any $d\,|\,(n,k)$.
From (3.16) and (3.17), we have

(3.18) $$\# P_{n,k} = \sum_{d|(n,k)} \psi(n/d, k/d),$$

which gives us the equality (3.14), becauce of (3.15). q.e.d.

From Lemma 3.2, we have immediately the total number of k-circular partitions of n in the following

Lemma 3.3. For any positive integer n and k, $1\leqq k\leqq n$, we have

(3.19) $$\# C_{n,k} = \sum_{d|(n,k)} \psi(n/d, k/d)/(k/d).$$

At this stage it is useful to appeal to the special property of the famous Möbius function which is defined

(3.20) $$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } d \text{ is not square-free,} \\ (-1)^{\nu} & \text{if } d \text{ is a product of } \nu \text{ distinct prime numbers,}\end{cases}$$

for any positive integer d. IN fact we observe, as in the classical use of the Möbius function, the following theorem.

Theorem 3.1. For any positive integer n and k, $1\leqq k\leqq n$, we have the equality

(3.21) $$\psi(n,k) = \sum_{d|(n,k)} \mu(d)\binom{n/d-1}{k/d-1}.$$

Now we shall first give the number of reverberation cycles with all the possible lengths under each of two operators $L_{\bar{\alpha}_0}$ and L_{α_0}, by using the results given above. First of all we shall explain the correspondence between state configuration(s) and an ordered partition. As a conventional notation of any n-stste configuration $\delta=(\delta_{n-1},\cdots,\delta_0)$ we abbreviate the same consecutuve states in its composition, by use of exponents ; for example, a 6-state configuration (1 ,1 , 1 , 0, 0, 1) is written by $(1^3, 0^2, 1^1)$. In this way any n-state configuration δ can be written

$$\delta = (\delta^{t_1}, \bar{\delta}^{t_2}, \cdots, \delta^{t_{k-1}}, \bar{\delta}^{t_k}) \quad \text{for } k = \text{even} \tag{3.22}$$

or

$$\delta = (\delta^{t_1}, \bar{\delta}^{t_2}, \cdots, \bar{\delta}^{t_{k-1}}, \delta^{t_k}) \quad \text{for } k = \text{odd}, \tag{3.23}$$

where $\delta = 1$ or 0, $\bar{\delta} \equiv 1-\delta$, $\sum_{i=1}^{k} t_i = n$ and $t_i \geqq 1$ for $i = 1, 2, \cdots, k$. Now the k-dimensional vector $(t_1, t_2, \ldots, t_k)$ defined just now forms a k-ordered partition of n and is uniquely determined for each assighned n-state configuration. Such a k-dimentional vector is called to be a k-ordered partition associated with n-state configuration. It is evident that for any k-ordered partition $(t_1, t_2, \ldots, t_k)$ there exist exactly two n-state configurations with each of which the partition $(t_1, t_2, \ldots, t_k)$ is associated, and these two statw configurations are mutually conjugate.

In what follows we denote by $N_L^n(R)$ the number of reverberation cycles with length R under the operator L defined in statae configuration space X_n.

3.1. The operator $L_{\bar{\alpha}_0}$

In this subsection we shall confine our discussion to the operator $L_{\bar{\alpha}_0}$.

Under the operator $L_{\bar{\alpha}_0}$ we have immediately the following

Corollary 3.1. For any n-state configuration δ in X_n, a reverberation cycle containing δ includes also its conjugate n-state configurarion $\bar{\delta}$ in X_n.

Lemma 3.4. For any assigned reverberation cycle there exists an odd integer k uniquely determined such that (i) there exists k-ordered partition $(t_1, t_2, \ldots, t_k)$ associated with some n-state configuration in the reverberation cycle, (ii) the set of all the ordered partitions associated with n-state configurations in the reverberation cycle consists of the set $E(t_1, t_2, \ldots, t_k)$, where $E(t_1, t_2, \ldots t_k)$ denotes the equivalent class of k-ordered partitions containing the ordered partition $(t_1, t_2, \ldots, t_k)$ given (i) and the set of (k+1) -ordered partitions.

Proof. For any reverberation cycle, we may and we shall note that there exists some n-state configuration δ which associates with a k-ordered partition of n, denoted by $(t_1, t_2, \ldots, t_k)$, with an odd integer k. This is due to the fact that in general, for any even integer ℓ and any n-state configuration $\delta = (\delta^{s_1}, \bar{\delta}^{s_2}, \ldots, \delta^{s_{\ell-1}}, \bar{\delta}^{s_\ell})$ where $\delta = 1$ or 0, n-state configuration $L_{\bar{\alpha}_0}^{s_\ell}(\delta)$ associated with $(\ell-1)$-ordered partition $(s_1+s_\ell, s_2, \ldots, s_{\ell-1})$.

In connection with $(t_1, t_2, \ldots, t_k)$, let us devide the set of 2n integers $I = \{1, 2, \ldots, 2n\}$ into the mutually disjoint sets

(3.24) $$I^{(1)} = \{ i \ ; \ i = \sum_{j=\alpha}^{k} t_j, \ \sum_{j=\alpha}^{k} t_j + n, \ \alpha = 1, 2, \cdots, k \}$$

and

(3.25) $$I^{(2)} = \{ i \ ; \ i \in I, \ i \notin I^{(1)} \},$$

Since the set of all the n-state configurations belonging to the reverberation cycle is written by $\{ L^{i}_{\bar{\alpha}_0}(\delta); i \in I \}$, in view of the definition of the operator $L_{\bar{\alpha}_0}$, the set of all the ordered partitions associated with each state configuration in $\{ L^{i}_{\bar{\alpha}_0}(\delta); i \in I^{(1)} \}$ is equal to the set $E(t_1,t_2,\ldots,t_k)$, while any n-state configuration in $\{ L^{i}_{\bar{\alpha}_0}(\delta); i \in I^{(2)} \}$ associates with a (k+1)-ordered partition of n. q.e.d.

The converse assertion of Lemma 3.4 is given by the following

Lemma 3.5. For any odd integer k and any equivalent class $E(t_1,t_2,\ldots,t_k)$ there exists an reverberation cycle uniquely determined which has an n-state configuration, associating with the k-ordered partition $(t_1,t_2,\ldots,t_k)$.

Proof. The set of the n-state configurations in X_n, with each of which k-ordered partition $(t_1,t_2,\ldots,t_k)$ is associated, consists of the following two state configurations.

(3.26) $$(1^{t_1}, \ 0^{t_2}, \ \cdots, \ 0^{t_{k-1}}, \ 1^{t_k})$$

and

(3.27) $$(0^{t_1}, \ 1^{t_2}, \ \cdots, \ 1^{t_{k-1}}, \ 0^{t_k}).$$

On the other hand, in view of Corollary 3.1, these two n-state configurations belong to the same reverberation cycle, which implies, by Lemma 3.4, that for any assigned equivalent class $E(t_1,t_2,\ldots,t_k)$ with an odd integer k there exists one and only one reverberation cycle. q.e.d.

Now the length of a RVC is defined by the number of all the different n-state configurations belonging to the RVC. We observe

Lemma 3.6. For any odd integer k and any k-ordered partition $(t_1,t_2,\ldots,t_k)$ of n, for which $\# E(t_1,t_2,\ldots,t_k)=j$ holds, the length of reverberation cycle corresponding to $E(t_1,t_3,\ldots,t_k)$ is given by $\frac{2n}{d}$, where $dj=k$.

Proof. For any odd integer k and any k-ordered partition $(t_1,t_2,\ldots,t_k)$, for which $\# E(t_1,t_2,\ldots,t_k)=j$ holds, let us denote an n-state configuration which $(t_1,r_2,\ldots,t_k)$ is associated by

(3.26) $$\delta = (\delta^{t_1}, \ \delta^{t_2}, \ \cdots, \ \bar{\delta}^{t_{k-1}}, \ \delta^{t_k}).$$

From Lemma 3.1, we have the equation

(3.27) $$(t_1, t_2, \cdots, t_k) = (t_1, t_2, \cdots, t_j)^{d},$$

with $jd = k$, which implies,

(3.28) $\quad \delta = (A\bar{A})^m A,$

(3.29) $\quad A = (\delta^{t_1}, \bar{\delta}^{t_2}, \ldots, \bar{\delta}^{t_{j-1}}, \delta^{t_j})$

where $m = \frac{d-1}{2}$, because both of d and j are odd integers. Hence we have

(3.30) $\quad \angle_{x_0}^{R}(\delta) = \bar{A}(A\bar{A})^m = (\bar{A}A)^m \bar{A}$

(3.31) $\quad \angle_{x_0}^{2R}(\delta) = A(\bar{A}A)^m = (A\bar{A})^m A = \delta$

(3.32) $\quad \angle_{x_0}^{R+\ell}(\delta) = \overline{\angle_{x_0}^{\ell}(\delta)},$

where $R = \sum_{i=1}^{j} t_i = \frac{n}{d}$ and $\ell < \frac{n}{d}$. On the other hand, since $\# E(t_1, t_2, \ldots, t_j) = j$, we have

(3.34) $\quad \delta^i(t_1, t_2, \ldots, t_j) \neq (t_1, t_2, \ldots, t_j),$

for $i=1,2,\ldots,j-1$, which implies, in view of (3.32), that we have

(3.34) $\quad \angle_{x_0}^{\ell}(\delta) \neq \delta,$

for $\ell = 1,2,\ldots,2\frac{n}{d}-1$. (3.31) and (3.34) give us that the length of RVC is equal to $2\frac{n}{d}$. q.e.d.

Lemma 3.7. The set of all the possible lengths of reverberation cycles is given by

(3.35) $\quad \{2\frac{n}{d}\ ;\ d$ is an odd integer such that $d \mid n\}$.

Proof. According to Lemma 3.1and Lemma 3.6, for any odd integer k and $d \mid (n,k)$, the length of RVC corresponding to $E(t_1, t_2, \ldots, t_k)$, for which $\# E(t_1, t_2, \ldots, t_k) = \frac{k}{d}$ holds, is given by the set

(3.36) $\quad \{2\frac{n}{d}\ \ \forall k, \forall d \mid (n,k)$ s.t. k is an odd integer, $1 \leq k \leq n\}$,

which is nothing dut the set (3.35). q.e.d.

After these preparations we reach to the final result of this subsection which read :

Theorem 3.2. For any odd integer d such that $d \mid n$, the number of reverberation cycles with the length $2R = 2\frac{n}{d}$ is given by

(3.37) $\quad N_{x_0}^{n}(2R) = \frac{1}{2R} \sum_{\substack{d' \mid R \\ d':\ odd}} \mu(d') 2^{R/d'}.$

Proof. Because of Lemma 3.6, for an odd integer d such that $d \mid n$, the set of all the equivalent classes of ordered partitions to each of which a reverberation cycle with length $2\frac{n}{d}$ coresponds, is given by

(3.38) $\quad D = \{E(t_1, t_2, \ldots, t_j)\ ;\ \sum_{i=1}^{j} t_i = k,\ \# E(t_1, t_2, \ldots, t_j) = j,$

j is an odd integer s.t. $1 \leqq j \leqq R \}$

with $R=\frac{n}{d}$. In view of Lemma 3.3, we have

$$(3.39) \qquad \# D = \sum_{\substack{1 \leqq j \leqq R \\ j:\text{odd}}} \psi(R,j)/j ,$$

which implies, from Theorem 3.1,

$$(3.40) \qquad \# D = \sum_{\substack{1 \leqq j \leqq R \\ j:\text{odd}}} \sum_{d'|(R,j)} \mu(d') \binom{R/d' - 1}{j/d' - 1} \Big/ j = \sum_{\substack{d'|R \\ d':\text{odd}}} \mu(d') \frac{1}{R} \sum_{l=1}^{\left[\frac{R/d'-1}{2}\right]} \binom{R/d'}{2l+1}$$

$$= \sum_{\substack{d'|R \\ d':\text{odd}}} \mu(d') \frac{1}{R} 2^{R/d'-1} = \frac{1}{2R} \sum_{\substack{d'|R \\ d':\text{odd}}} \mu(d') 2^{R/d'} .$$

But Lemma 3.4 and 3.5 show that RVC with length $2R=2\frac{n}{d}$, which, in view of (3.40), gives us (3.37), as we were to prove. q.e.d.

Corollary 3.2. We have the following equation,

$$(3.41) \qquad 2^{n} = \sum_{\substack{d|n \\ d:\text{odd}}} 2\frac{n}{d} N^{n}_{L_{\bar{\alpha}_0}} (2\frac{n}{d}),$$

where $N^{n}_{L_{\bar{\alpha}_0}}(2\frac{n}{d})$ is given by (3.37).

3.2. The operator L_{α_0}

In this subsection we confine our discussion to the operator L_{α_0}.

Quite similarly as the case of the operator $L_{\bar{\alpha}_0}$, so we omit the proofs.

Lemma 3.8. For any assigned reverberation cycle there exists an even integer k uniquely determined such that (i) there exists a k-ordered partition $(t_1,t_2,\ldots t_k)$ associated with an n-state configuration in the reverberation cycle, (ii) the set of all the ordered partitions associated with n-state configurations in the reverberation cycle consists of the set $E(t_1,t_2,\ldots,t_k)$ and the set of (k+1)-ordered partitions of n.

Lemma 3.9. For any even integer k and any equivalent class $E(t_1,t_2,\ldots,t_k)$, (i) if $\# E(t_1,t_2,\ldots,t_k)$ is an odd integer, then there exists an reverberation cycle uniquely determined such that an n-state configuration in the reverberation cycle associats with a k-ordered partition in the $(t_1,t_2,\ldots,t_k)$, while (ii) if $\# E(t_1,t_2,\ldots,t_k)$ is an even integer, then there exists two reverberation cycles such that these are mutually conjugate reverberation cycles and each of these reverberation cycle contains an n-state configuration which associates with k-ordered partition $(t_1,t_2,\ldots,t_k)$ of n.

Lemma 3.10. For any even integer k, $d|(n,k)$ and any k-ordered partition $(t_1,t_2,\ldots,t_k)$, for which $\# E(t_1,t_2,\ldots,t_k)=j$ with $jd=k$ holds, the length R

of reverberation cycle corresponding to $E(t_1,t_2,\ldots,t_k)$ is given as follows:
(i) for the case that j is an odd integer, $R = 2\frac{n}{d}$,
(ii) for the case that j is an even integer, $R = \frac{n}{d}$

Lemma 3.11. The set of all the possible lengths of reverberation cycle is given by

(3.42) $$\left\{ \frac{n}{d} \;;\; d \mid n \right\}.$$

Theorem 3.3. For any positive integer d such that $d \mid n$, the number of reverberation cycles with length $R=\frac{n}{d}$ is given by
(i) for any odd integer $R \geqq 3$,

(3.43) $$N^{n}_{L_{\alpha_0}}(R) = \frac{1}{R}\sum_{\substack{d' \mid R \\ d':\,odd}} \{u(d')(2^{R/d'}-2)\}$$

while

(3.44) $$N^{n}_{L_{\alpha_0}}(1) = 2$$

(ii) for any even integer R,

(3.45) $$N^{n}_{L_{\alpha_0}}(R) = \frac{2}{R}\sum_{\substack{d' \mid R \\ d':\,odd}} u(d')\{2^{R/d'-1}-2\} + \frac{2}{R}\sum_{\substack{d' \mid R \\ d':\,even}} u(d')\{2^{R/d'}-2\} + \frac{1}{R}\sum_{\substack{d' \mid \frac{R}{2} \\ d':\,odd}} u(d')2^{R/2d'}.$$

3.3 The other operators belonging to the $\mathcal{L}_{\omega}$ set

In view of Theorem 1.1, we should notice that, in contrast with L_{α_0} and $L_{\bar{\alpha}_0}$, there exists their respective subsets of n-state configurations, each of which does not belong to an RVC for each L_{α_ℓ} and $L_{\bar{\alpha}_\ell}$ when $1 \leqq \ell \leqq n-1$, but each of which reaches to an n-state configuration belonging to some RVC after finite applications of each of these operators. In short, each of these n-state configurations is transient. In spite of the existence of these transient state configurations, it will be shown in this subsection that so far as the n-state configurations belonging to the RVC are concerned, the number of RVC's in each of L_{α_ℓ} and $L_{\bar{\alpha}_\ell}$ can be reduced to those of L_{α_0} and $L_{\bar{\alpha}_0}$ which were already given in Subsection 3.1 and 3.2.

Theorem 3.4. For any positive integer $d \mid (n-\ell)$, we have

(3.46) $$N^{n}_{L_{\alpha_\ell}}\left(\frac{n-\ell}{d}\right) = N^{n-\ell}_{L_{\alpha_0}}\left(\frac{n-\ell}{d}\right),$$

(3.47) $$N^{n}_{L_{\bar{\alpha}_\ell}}\left(2\frac{n-\ell}{d}\right) = \begin{cases} N^{n-\ell}_{L_{\bar{\alpha}_0}}\left(2\frac{n-\ell}{d}\right) & \text{for an odd integer } d \\ 0 & \text{for an even integer } d, \end{cases}$$

for $\ell = 1,2,\ldots,n-1$.

The proof is omitted.

It may be noted that an n-state configuration in X_n can attein to some RVC at least ℓ applications of L_{α_ℓ} ($L_{\overline{\alpha}_\ell}$) ($\ell = 1,2,\ldots,n-1$), i.e., the length of transient phrase is at most ℓ.

On the other hand it is immediately to observe that any n-state configuration in X_n except $(1,1,\ldots,1)$ ($(0,0,\ldots,0)$) belongs to a transient phrase under the operator L_ω($L_{\overline{\omega}}$), while the n-state configuration $(1,1,\ldots,1)$ ($(0,0,\ldots,0)$) in X_n constitutes a RVC with length one under the operator L_ω ($L_{\overline{\omega}}$).

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西尾：オペレータのクラスα－ωに属する threshold function は特殊なものですね。定数でみれば どんなものになっているのか。

山口：α－ωに属するオペレータに対応する結合係数ですね。それは……

西尾：すぐにわかりますね。i番目だけとってきて，それだけ，大きい係数にして．

山口：ええ．それを非常に大きくして。conjugate にする場合には，θを負の値にして．

西尾：ひっくり返すわけですね。それに目をつけたのは何故かということ。まあ．いきなりむずかしいことは全部はやれないから。

甘利：一番最後の結論をもう一度いうと，α－ωという特殊オペレータで表わされる回路を考える。そのとき．状態は 2^n 個あるのだけれども，これらの状態は．α－ωによって．いくつかのサイクルをつくる。そこで長さが $(n-l)/d$ のサイクルが何個あるかをしらべてみると，その数は．その漸化式をみたす。そういうことですか。

山口：はい．

西尾：それは枝がはえていてもいいのですね。

山口：こっち（$L_{\alpha_l}, L_{\bar{\alpha}_l}, l=1,2,\cdots,n-1$）は枝がはえて

います。こちらのオペレータは…。

甘利：　え>，0になるときだけたまたま枝がなくなるわけですね。

山口：　はい，すべてがサイクルになってしまいます。

甘利：　しかし，西尾さんのおっしゃったとうりだけれどもそのL_{d_ℓ}というのは，Threshold function のなかできわめて特殊な形をしていますね。またグラフの話がでてきましたが，あれはうまくゆくのですか。

山口：　いろいろとりあつかってゆくと，結局はもとの関数方程式にもどってしまいます。

甘利：　この話の結論というのは，例えば，$n=10$で，L_{d_5}でいいですが，サイクルの数はどれぐらいあるのですか。

山口：　$n=10$で，$L_{\bar{d}_0}$の場合を考えてみます。例えば，RVCの長さが，4であるものは，

$$N^{10}_{L_{\bar{d}_0}}(4)=\frac{1}{4}\sum_{\substack{d'|2 \\ d':\text{odd}}}\mu(d')2^{2/d'}=\frac{1}{4}(4)=1.$$

従って2^{10}個あるstate configurationのなかで長さが4のRVCの数は1個ということになります。

甘利：'最大周期はどの位になりますか。

山口：　それは，周期の長さは$2\frac{n}{d}$で与えられますから，$d=1$とおいたとき，すなわち$2n$です。

西尾：　2n はでるのですね．これで．

甘利：　そうですね．納得いきます．直感的にも…。しかし一般の回路だと最大周期は 2n より大きくなるでしょう。

西尾：　こう考えたらどうですか。いまの甘利さんの質問に関係あるのですが，結局 Caianiello のモデルで 1-element の場合は，McCulloch-Pitts のモデルの (n+1)-elements の回路と同じで，それの結線を限ったものが $L_{\alpha-\omega}$ になるわけです。だからその回路の最大周期ということですね。

甘利：　今のモデルは等価的には，不応期のない神経素子をリング状にならべておいて，途中は1本づつ矢印を出してゆく。つまり単なる shift registor です．ただ最後だけが，他の n 個からすべて結線がある．でそれは，一般の n 素子からなる閾素子回路のごく特殊な場合ですね．で，いまの $L_{\alpha\ell}$, $L_{\bar{\alpha}\ell}$ は．そのなかでも特に ℓ 番目だけから線をひっぱってくるという更に特殊な場合になっている。そしてこれらの場合は全部話がかたづいている。それでは，この論文の最初のモデルの場合には，最大どれ位の周期の RVC が出現しうるかということです．

西尾：　しかしそれをきっちりやるというのはむつかしいかもしれませんね．

甘利：　そうですね．

西尾：　不等式ではさんで、このあたりに最大周期があるというのならできるかもしれないけれども。この場合限られてはいるけれども、それについてはすべてが解けているわけですね。それと、一般の回路で、最大周期だけを求める場合とどちらがやさしいか。後者は不等式でおさえるということぐらいになるでしょう。そんな意味からすれば、ここの話はきっちりしていて，結果はすっきりしています。

甘利：　グラフができたり整数論的なものができたり楽しいですね。話は全然変るけれども，いまのδを0あるいは1をとるということにしないで，例えば0と1との間の実数とすると，そして，例の1[・]という関数もこのままだと，0または1という値しかとらないから，それもある連続関数とすると，話はむずかしくなりますか。例えば，ランダムに結合した神経回路の活動度のようなものですね。1の個数のパーセントと相対不応期を入れると，同じ形の式が得られます。ただδが0と1の間の実数になるというだけです。もとの差分方程式にもどせば高階の差分方程式ですね。

西尾：　この講演に関して，これは finite な state のオートノーマスな系の場合ですね。それの状態遷移図について，サイクルに枝があるものの集合と，それらの characterization ということですね。僕らは，それをセル・オートマトンでやっている

のですが．これは．~~関~~関数の場合で．むずかしいということは．十分身にしみています．これだけできたらきれいですよ。